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THE CONSUMPTION RATE OF INTEREST AND THE NUMBERS EFFECT*

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This paper explores how the number of people affected can influence the determination of the Consumption Rate of Interest (CRI). To date, the current literature has focused only on numbers in an intergenerational context, via a concern with population size. These papers are summarized showing the commonality of the various approaches. From there, the role of numbers in an intragenerational setting is developed. The main result is that the "numbers effect" proper (the number of uncompensated losers from a project) raises the value of the CRI.

I. INTRODUCTION

A recent paper published in this journal advocated incorporating the "numbers effect" (trying to reduce the number of uncompensated losers from a public project) into project appraisal (see Brent [1991b]). In the context of the Shadow Wage Rate, it was as a concern for employment that the "numbers effect" would become operational. However, just as the other policy objectives efficiency and distribution have pervasive influence in social decision-making, it is important to recognize that the numbers effect has a role to play in other parts of project appraisal methodology. It is particularly important in the theory underlying the concept of the Social Discount Rate (SDR). The area that is potentially most relevant is that of the Social Time Preference rate, which in project appraisal is called the "Consumption Rate of Interest" (CRI). The purpose of this article is to summarize and interpret how the numbers effect has been

* This paper covers a part of my survey of the Social Discount Rate that appears as Brent [1991a]. I wish to thank an anonymous referee for helping me restructure this version.

1 When consumption is the numeraire, as in the UNIDO [1972] approach, the CRI is the SDR. When public income (investment) is the numeraire, as in the Little and Mirrless [1974] and Squire and van der Tak [1975] approaches, the CRI is still worth knowing as it helps determine the shadow price of government income.
treated in the CRI literature in an intergenerational context and explain how it can be extended to an intragenerational setting.\textsuperscript{2}

II. NUMBERS IN AN INTERGENERATIONAL CONTEXT

The Consumption Rate of Interest $i_C$ can be defined as the rate of fall in the value of consumption over time:

\begin{equation}
    i_C = -\frac{\dot{W}_C}{W_C}
\end{equation}

where $C$ is aggregate consumption, $W$ is social welfare, and a dot above a variable signifies the derivative with respect to time.

In an individualistic framework, $W$ depends on the utilities of individuals in society, $U(c)$, with the consumption level $c$ being the only argument. The project appraisal literature has uniformly adopted the same set of assumptions for $U(c)$. Individuals are to have identical taste functions, such that they have different utilities only when their incomes differ. For then their consumption levels would not be the same. Marginal utilities, $U_c$, are taken to be isoelastic functions of individual consumption levels:

\begin{equation}
    U_c = c^{-\varepsilon} \quad (\varepsilon \geq 0)
\end{equation}

where $\varepsilon$ is the elasticity parameter.

The fundamental problem in deriving the CRI is how to use (2), based on individual consumption levels $c$, to derive the social marginal utility of consumption, $W_C$, that appears in the definition of $i_C$ and is based on aggregate consumption levels, $C$. If there were only one individual per generation, $c$ would equal $C$. $W$ would equal $U(c)$, and using (2):

$$W_C = C^{-\varepsilon}.$$ 

Consequently, the CRI would be:

\begin{equation}
    i_C = -\frac{\dot{W}_C}{W_C} = \varepsilon C^{-\varepsilon-1} \dot{C} / C = \varepsilon \dot{C} / C = \varepsilon G
\end{equation}

where $G$ is the growth rate of aggregate consumption. Clearly this Robinson Crusoe case is not very useful. We shall proceed in two stages. In the first we shall assume that consumption is equally distributed, and in the second deal with unequal distribution.

\textsuperscript{2} A full statement of all the derivations, and an examination of the role of the pure time preference rate, is presented in Brent [1991a].

II. A. The CRI When Consumption Is Equally Distributed

When society consists of more than one individual, the assumption is usually made that $C$ is shared equally among the $P$ individuals. This means that:

\begin{equation}
    c = C/P.
\end{equation}

Layard [1972] considers two formulations of this problem:

\textsc{Case 1:} $W$ depends only on consumption per head, $c$. The chain rule can be used to obtain $W_C$ since this is what appears in the definition of $i_C$:

\begin{equation}
    W_C = W_c \partial c / \partial C = W_c P^{-1}.
\end{equation}

$W_c$ is identified with $U_c$ from (2) and, using (4), eq. (5) becomes:

\begin{equation}
    W_C = C / P^{-\varepsilon} P^{-1}.
\end{equation}

The CRI that derives from this is:

\begin{equation}
    i_C = \varepsilon g + p
\end{equation}

where $g$ is the rate of growth of per capita consumption and $p$ is the rate of population growth.

\textsc{Case 2:} $W$ depends on population times the marginal welfare from $C$. The right-hand side of (6) is to be multiplied by $P$ to obtain $W_C$, i.e.,

\begin{equation}
    W_C = C / P^{-\varepsilon}.
\end{equation}

The CRI now takes the form:

\begin{equation}
    i_C = \varepsilon g.
\end{equation}

Cases 1 and 2 presented above can best be understood in terms of Feldstein’s [1965] general formulation of the SWF. The utilitarian version of the SWF defines $W$ as the sum of individual utilities:

\begin{equation}
    W = \int_0^P U(c) dc.
\end{equation}

Feldstein, however, argues that: “We may not wish to assert that a society which doubles its population while keeping constant a ‘low’ level of per capita income has necessarily doubled social utility.” He therefore recommends as a replacement:

\begin{equation}
    W = \Phi(P) \int_0^P U(c) dc \quad (P \leq 1).
\end{equation}
When consumption is equally distributed, (11) is to take the form:

\[ W = P^a U(c) \quad (0 \leq \alpha \leq 1). \]

The derivative of this with respect to aggregate consumption would be:

\[ W_C = P^a U_c P^{-1} = P^{a-1} U_c. \quad (12) \]

From this we can see that Layard’s two cases correspond to the two extreme values. With \( \alpha = 0 \) we get eq. (5) of Case 1, and with \( \alpha = 1 \) eq. (8) of Case 2.

Feldstein’s version accommodates both population effects and is therefore more satisfactory than either of Layard’s extreme cases. Feldstein’s version uses (12) with (2) to produce:

\[ i_C = (1 - \alpha) p + \varepsilon g. \quad (13) \]

There are, to my knowledge, no direct applications of (13) in the project appraisal literature, apart from the extreme cases. Nonetheless, the next sub-section shows how the intermediate cases for \( \alpha \) can be incorporated in an indirect fashion.

**II. B. The CRI When Consumption Is Unequally Distributed**

The previous formulations were based on there being either one individual or \( P \) identical individuals. When there are many individuals with unequally distributed incomes, one needs to ask the question, whose consumption is being affected by the public project? This question can be paraphrased to ask whom is to represent society in deciding the rate at which \( W_C \) is to decline over time? Squire and van der Tak (1975) – hereafter S&T – chose the person at the average level of consumption \( \bar{c} \) to be the representative. It would seem that, as they are stating that \( W \) depends only on \( \bar{c} \), S&T are in a Case 1 situation where \( W \) depends only on \( c \), consumption per head. However, they end up with Case 2’s eq. (9). They take a different route to get there. Rather than use an \( i_C \) based on \( W_C \), they choose to define \( i_C \) in terms of \( W_e \). The rate of fall over time of \( W_e \) is established from the outset and does not have to be derived as in eq. (8).

S&T’s version used one individual to represent the generation no matter the size of the generation. As an alternative, it would seem sensible to find out how many persons are in the position of the average \( P(\bar{c}) \), and to weight \( W_e \) by the share of \( P(\bar{c}) \) in the total population. In this case:

\[ W_C = \frac{P(\bar{c})W_e}{P} = f(\bar{c})W_e \quad (14) \]

where \( f(\bar{c}) = P(\bar{c})/P \).

The assumptions behind (14) are worth making explicit. A unit of aggregate consumption \( C \), whose value over time is being evaluated, is being shared out equally among the total population. So each person in the position of the average receives \( 1/P \), i.e., \( \bar{c} = \frac{BC}{PC} = P^{-1} \). The chain rule sets:

\[ W_C = W_p P^{-1}. \quad (15) \]

Like Case 2, \( W \) will depend on population times \( W_C \). But, this time the relevant population is \( P(\bar{c}) \) and not \( P \). The right-hand side of (15) is to be multiplied by \( P(\bar{c}) \) and this forms (14).

The corresponding CRI to (14) is, with \( \bar{c} \) used instead of \( c \) in (2):

\[ i_C = \varepsilon g - \phi \quad (16) \]

with \( \phi \) the growth rate of \( f(\bar{c}) \).

As \( \phi = p(\bar{c}) - p \), (16) can be rewritten as:

\[ i_C = \varepsilon g + p - p(\bar{c}). \quad (17) \]

This alternative version can be called the generalized S&T case. It is in some sense the “correct” S&T version. For they do not use (10) as the utilitarian SWF, but instead use the group version:

\[ W = \int U(c)f(c)dc \quad (18) \]

where \( f(c) \) is defined as “the density function of the distribution of consumption”. The derivative of \( W \) with respect to \( \bar{c} \) would be:

\[ W_e = U(\bar{c})f(\bar{c}) + f(\bar{c})U_e. \quad (19) \]

If the first term could be ignored (for example, by some normalization procedure which sets \( U(\bar{c}) = 0 \), then (19) is turned into (14). Eq. (17) would then be exactly the CRI.

Eq. (17) shows that the generalized S&T version incorporates the intermediate cases for \( \alpha \) that existed in Feldstein’s version. When \( P(\bar{c}) \) grows at the same rate as \( p, \bar{c}(\bar{c}) \) equals \( p \), and (17) reproduces Case 2. If \( P(\bar{c}) \) does not change over time, \( p(\bar{c}) \) equals zero, and Case 1 appears. The main difference is that \( \alpha \) is a normative parameter while \( f(\bar{c}) \), and hence \( p(\bar{c}) \), can be measured more readily. Eq. (17) confirms a Feldstein theorem. His results do not require that consumption be equally distributed, but only that the relative frequency of each income class remains unchanged. When
relative frequencies remain unchanged, $\phi$ must be equal to zero. Hence Case 2 is re-established.

III. THE NUMBERS EFFECT AND THE CRI

In this section an alternative to the utilitarian SWF will be presented. Central to any discussion of the form of a SWF is the recognition that interpersonal comparisons of cardinal individual utilities must be involved. Ng [1981] has highlighted the fact that, under ordinality, multiplying utilities has the same effect as adding utilities, if a monotonic transformation (logarithmic) of individual utility indicators is allowed. There would thus be no point in distinguishing multiplication from addition as is intended in this section. A full defence of the numbers effect, and its role in project appraisal as a whole, is contained in Brent [1984, 1986, 1990, and 1991b]. Here we present only sufficient introduction to the "numbers effect" to make its extension to the CRI intelligible.

The setting for the empirical SWF was intragenerational. A set of decisions had been made within a cost-benefit framework where the assumed goal was welfare maximization. The SWF uncovered by Brent [1984] had three elements: efficiency, distribution, and the numbers effect. To facilitate comparability with the previous discussion (a) efficiency will be gauged by the sum of utilities expression on the right hand side of eq. (10), and (b) the consumption distribution will be assumed to remain unchanged over time. It is only the third element, the numbers effect, that will be examined further in this section.

The welfare economic base behind cost-benefit analysis and project appraisal rests on the nature of Pareto optimality. When the sum of utilities is positive the gainers can compensate the losers and everyone can be made better off. Public projects, especially in developing countries, rarely are accompanied by such compensation. In which case there will be some uncompensated losers, $N$. The SWF under consideration is:

$$ W = W \left[ N, \int U(c)dc \right] \quad (W_0 \leq 0). \quad (20) $$

In the actual context in which the three objective SWF was observed (railway closure decisions in the U.K.), $N$ entered additively and net benefits were in a per capita form. That is, the SWF took the form:

$$ W = a_N N + U(c) \quad (a_N \leq 0) \quad (21) $$

where $a_N$ is the weight to the numbers effect.

In the railway closure context, $N$'s effect was entirely intragenerational. Its effect can be transformed into an intergenerational one through the specification of $a_N$. Since Pareto optimality is the welfare base behind (34), $a_N$ can be interpreted as the weight to departures from unanimity. Unanimity would be where all $P$ persons could be compensated. Thus the weight to any one individual causing a departure from unanimity would be $1/P$, if all individuals are weighted equally. Eq. (21) in an intergenerational setting would be:

$$ W = -(1/P)N + U(c). \quad (22) $$

As the SWF was in a per capita form, it seems appropriate to follow S&T's procedure for deriving the CRI. The derivative of $W$ with respect to per capita consumption is:

$$ W_c = -(1/P)N_c + U_c = -f(N_c) + c^{-e} \quad (23) $$

where $f(N_c)$ is defined as $N_c/P$, and $U_c$ is given by (2). A per capita definition of the CRI turns $i_C$ into:

$$ i_C = \frac{f(N_c)}{f(N_c) + \epsilon c^{-e}} \quad (24) $$

The impact of the numbers effect on the CRI can be seen from the partial derivative: $\partial i_C/\partial f(N_c)$. Its sign depends crucially on the rate of growth of $f(N_c)$. As long as this is non-negative (as is the experience for most LDCs) an increase in $N$ (from changing $c$) will raise the CRI.

That the numbers effect will raise the CRI can easily be seen by considering an important special case. A standard argument for ignoring income redistribution in project appraisal was that, over time, the full set of development projects would cause adverse effects to be neutralized by positive effects. In which case everyone would gain and there would be no uncompensated losers. Unanimity would be achieved in the long run. Although the empirical validity of this line of reasoning is lacking in LDCs (and elsewhere), this can be set up as an objective of the planning authority. We can therefore consider the "steady state" CRI given by $f(N_c) = 0$. It is immediate from (24) that $f(N_c)$ raises $i_C$. The economic interpretation of this is straightforward. The CRI compares the preference for consumption today with the preference for consumption in the future. Preference for consumption in any period is higher the

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3 Important differences between the following discussion and Brent [1984] are that, in the latter, distributional weights in cash and kind were used, and weights were discrete rather than continuous like those in (2).

4 The main role of $a_N$ in an intragenerational context is to convert $N$ into dollars in order to be commensurate with the weighted efficiency benefits and cost.

5 This partial derivative is derived and discussed in full in Brent [1991a].
lower the level of social welfare. Because there will be uncompensated losers today, welfare will be lower today and there will be a greater preference for consumption today. Hence, the CRI will be higher.  

In line with S&T, \( c \) will be replaced by \( \tilde{c} \). A person at the average level of consumption is again to represent society. But, this time an allowance is being made for the fact that as this individual is being made better off by an increase in \( c \), other individuals (the uncompensated losers) are being adversely affected. This is the meaning of (21) on which (24) was based.

To help provide some insights into how \( ic \) is affected by the interaction between \( f(N_c) \) and the other parameters, and to show how (24) can actually be calculated, Table 1 shows how \( ic \) varies with the main parameters. Only the crucial cases for \( f(N_c) \) and \( f(N_c) \) are presented. They are meant to illustrate the full range of possibilities, and are not necessarily the values that are the most likely. \( \tilde{c} \) will take S&T's recommended value of unity and \( g \) will always be set at 0.01. The product \( eg \), which is S&T's CRI, is therefore equal to 0.01 throughout the table. There are three sections in the table, corresponding to low, medium, and high levels of \( c \). These levels are to relate to some base figure, which for convenience is fixed at \( c = 1 \). With this base, \( c = .5 \) is the low level and \( c = 2 \) the high level. \( f(N_c) \) takes the values 0.2 and 0.8. \( \tilde{f}(N_c) = 0 \) is the steady state value. We consider also values that are above and below the steady state in proportion to the size of \( f(N_c) \).

Table 1 shows that there are two dimensions to the numbers effect as it relates to the CRI, i.e., how large is \( N \) today and how will it change in the future. When \( N_c \) is low relative to \( P, f(N_c) = 0.2, \tilde{c} \) rises with positive \( f(N_c) \) and falls with negative \( f(N_c) \). This is true for all levels of consumption. \( ic \) remains positive throughout. However, when \( f(N_c) \) is high, equal to 0.8, it may swamp the marginal utility of consumption effect thus making the current marginal welfare \( W_c \) negative. In these cases \( ic \) could become negative, signifying that any positive improvement in the future (via \( eg \) for example) would make the investment socially worth undertaking. The last line in the table shows the extreme case where both satisfaction from consumption today and in the future are negative, thus causing a positive \( ic \).

IV. SUMMARY AND CONCLUSIONS

Numbers have always been recognized to have an impact on the CRI. To date this has been viewed in an intergenerational setting, i.e., in terms of allowing for

\[ \text{population size. The contribution of this paper has been to add an intragenerational dimension. How numbers in all its guises affects the CRI can be summarized by referring to the five major versions covered in this paper that are listed in Table 2.} \]

<table>
<thead>
<tr>
<th>ic</th>
<th>( e^{g \phi} )</th>
<th>( f(N_c) )</th>
<th>( f(N_c) + e^{g \phi} )</th>
<th>( -f(N_c) + e^{g \phi} )</th>
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<tr>
<td>0.01</td>
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<td>0.000</td>
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<tr>
<td>0.01</td>
<td>0.020</td>
<td>0.2</td>
<td>+0.004</td>
<td>0.024</td>
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<tr>
<td>Low</td>
<td>0.01</td>
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<td>0.01</td>
<td>0.020</td>
<td>0.8</td>
<td>+0.016</td>
<td>0.036</td>
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<tr>
<td>Low</td>
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<td>0.8</td>
<td>-0.016</td>
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<tr>
<td>0.01</td>
<td>0.010</td>
<td>0.2</td>
<td>0.000</td>
<td>0.010</td>
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<tr>
<td>0.01</td>
<td>0.010</td>
<td>0.2</td>
<td>+0.004</td>
<td>0.014</td>
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<tr>
<td>Medium</td>
<td>0.01</td>
<td>0.010</td>
<td>0.8</td>
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<tr>
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<td>0.010</td>
<td>0.8</td>
<td>+0.016</td>
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<tr>
<th>ic</th>
<th>( \text{Equation} )</th>
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<tbody>
<tr>
<td>Layed Case 1</td>
<td>( ic = eg + p )</td>
<td>(7)</td>
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<tr>
<td>Layed Case 2 and S&amp;T</td>
<td>( ic = eg )</td>
<td>(9)</td>
</tr>
<tr>
<td>Feldstein</td>
<td>( ic = (1 - n)p + eg )</td>
<td>(13)</td>
</tr>
<tr>
<td>Generalized S&amp;T</td>
<td>( ic = eg - \phi_{eg} = eg - \phi )</td>
<td>(16)</td>
</tr>
<tr>
<td>Numbers Effect</td>
<td>( ic = \frac{f(N_c) + e^{g \phi}}{-f(N_c) + e^{g \phi}} )</td>
<td>(24)</td>
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The key is to focus on the denominator part of the \( ic = \frac{\tilde{W}_c}{W_c} \) definition. Anything that lowers \( \tilde{W}_c \), ceteris paribus, raises the CRI. The first three cases in the table deal with numbers in an intergenerational context. A higher population lowers
the amount of consumption available. Usually this will be sufficient to lower $W_C$ and hence raise the CRI. This can be seen by inspecting the $W_C$ specifications for these three cases. In eq. (8), $P$ unambiguously lowers $W_C$; in (6), $P$ will lower $W_C$ provided that $\varepsilon < 1$; and in (12), $P$ reduces $W_C$ if $\alpha < 1$.

Up to this point, numbers enter the analysis only as a determinant of per capita consumption, i.e., $C/P$. Numbers in an intragenerational context play a role by augmenting $W_C$, either in a multiplicative, or in an additive manner. While the endpoint of S&T was the same as for Layard Case 2, the starting point was very different. It introduced (but did not employ in the analysis) the idea that the marginal utility of consumption should be multiplied by the share of the population who are at the average level of consumption — see $f(\varepsilon)$ in eq. (14). The more people who are at the level of the average, the greater is $W_C$. This explains then why the CRI which works with this base, the "generalized S&T version", is lower when numbers are included. Its contribution is that it accommodates Feldstein's intermediate cases by replacing $\alpha$ with the measurable parameter $f(\varepsilon)$.

The CRI based on the "numbers effect" proper, i.e., the number of uncompensated losers, shares a common property with the generalized S&T version. That is, the relative size of one group is of social concern. But, this time it operates in reverse. The more uncompensated losers there are, the lower is the denominator of $IC$ (see eq. (23)) and the value of CRI was thereby increased. This explains the main new result of the paper, that the numbers effect will raise the CRI.

The other difference between the numbers effect CRI and the generalized S&T version is that the additive property makes it more likely that the CRI will be negative. $W_s$ is the difference between the positive marginal utility of consumption and the negative numbers effect (see eq. (23)). The sign of $W_s$ is not therefore unambiguously positive.

REFERENCES


